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Division algebras with integral elements

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Abstract. Pairing two elements of a given division algebra furnished with a multiplication rule leads to an algebra of higher dimension restricted by 8. This fact is used to obtain the roots of SO(4) and SP(2) from the roots ∓ 1 of SU(2) and the weights $\mp\frac{1}{2}$ of its spinor representation. The root lattice of SO(8) described by 24 integral quaternions are obtained by pairing two sets of roots of SP(2). The root system of F_4 is constructed in terms of 24 integral and 24 'half integral' quaternions. The root lattice of E_8 expressed as 240 integral octonions are obtained by pairing two sets of roots of F_4. Twenty four integral quaternions of SO(8) forming a discrete subgroup of SU(2) are shown to be the automorphism group of the root lattices of SO(8), F_4 and E_8. The roots of maximal subgroups SO(16), $E_7 \times SU(2)$, $E_6 \times SU(3)$, SU(9) and $SU(5) \times SU(5)$ of E_8 are identified with a simple method. Subsets of the discrete subgroup of SU(2) leaving maximal subgroups of E_8 are obtained. Constructions of E_8 root lattice with integral octonions in seven distinct ways are made. Magic squares of integral lattices of Goddard, Nahm, Olive, Ruegg and Schwimmer are derived. Possible physical applications are suggested.

1. Introduction

Supersymmetric Yang-Mills and superstring theories [1] in critical spacetime dimensions $d = 3$, 4, 6 and 10 attracted much attention regarding their relevance to four division algebras $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions or Cayley numbers) [2]. Their respective dimensions 1, 2, 4 and 8 equal $d-2$ physical modes corresponding to various transverse degrees of freedoms in critical spacetime dimensions.

Their norm groups and automorphism groups will be our special interest. The norm groups of division algebras are linear transformations of the components of an element of the algebra which preserve the quadratic norm invariant. The norm groups are the discrete group Z_2 for real numbers, U(1) for complex numbers, $SO(4) \approx SU(2) \times SU(2)$ for quaternions and SO(8) for octonions. The automorphism groups are the groups leaving the multiplication table of the imaginary units of the algebra invariant. They are Z_2 for complex numbers, SU(2) for quaternions and G_2 for octonions [3].

In this work we construct the root systems of the groups SU(2), $SU(2) \times SU(2)$, SO(8) and E_8 with integral elements of four division algebras associated with real

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numbers, complex numbers, quaternions and octonions respectively [4]. We start with ± 1, the units of real integers and half integers $\pm\frac{1}{2}$ representing respectively the non-zero roots and the weights of two-dimensional spinor representation of SU(2) and construct the remaining integral elements of complex numbers, quaternions and octonions. To do this we follow a well defined procedure 'pairing' two elements of a given algebra $[a, b]$ to define an element in an algebra of higher dimension. The by-product of this approach is the emergence of the root systems of SP(2) and F_4 associated with complex numbers and quaternions respectively. We obtain the root lattice of SO(8) generated by integral quaternions by pairing two sets of SP(2) roots expressed as integral and 'half-integral' complex numbers. Similarly the root lattice of E_8 generated by 240 integral octonions can be obtained by pairing two sets of roots of F_4 expressed as 24 integral quaternions and 24 'half-integral' quaternions.

The techniques we employ in this work may have an immediate application in the study of lattice construction of chiral fermionic strings, nicely discussed and reviewed by Schellekens and his collaborators [5]. Another relevant application may arise in a possible extension of orbifold compactification of the heterotic string with Abelian symmetries to orbifolds with non-Abelian symmetries [6] with regard to the fact that the root system of SO(8) corresponds to the binary tetrahedral group which we will discuss later.

The paper is organised as follows. In § 2 we introduce the definition of an integral elements of a division algebra and give the necessary machinary for further calculations. In § 3 we construct the root systems of SO(4), SP(2), SO(8) and F_4. The integral quaternions describing the root lattice of SO(8) form a discrete subgroup of SU(2) of order 24 called a binary tetrahedral group. It is shown that this group is the automorphism group of the root systems of SO(8) and F_4. When 24 short roots of F_4 multiplied by $\sqrt{2}$ and combined with 24 integral quaternions (long roots of F_4) it is shown that they form a discrete subgroup of SU(2) of order 48 called a binary octahedral group. In § 4 we find a way of pairing of two sets of the roots of F_4 to obtain the root lattice of E_8 expressed as integral octonions. As an intermediate step the root lattice of SO(16) is constructed with pairing two systems of roots of SO(9) and it is shown that the integral octonions describing SO(16) roots do not close under octonionic multiplications. This provides a hint for the construction of E_8 root lattice from two sets of roots of F_4 which lead to 240 integral octonions which are closed under multiplication. Action of the elements of the binary tetrahedral group on the root system of E_8 is investigated. Section 5 is devoted to branching of roots of E_8 under its maximal subgroups SO(16), $E_7\times$SU(2), $E_6\times$SU(3), SU(9) and SU(5)$\times$SU(5). A simple method is devised for the identification of these groups. Symmetries preserving the coset structures of E_8 with respect to its maximal subgroups are derived. The magic squares of Goddard, Nahm, Olive, Ruegg and Schwimmer (GNORS) [7] are constructed with the method described here. A decomposition of the roots of E_8 with respect to the subgroup $F_4\times G_2$ is also obtained. In § 6 we remark on the possible use of the binary tetrahedral group in the vertex construction of level one representation of affine F_4 and E_8 algebras. The possible use of the binary tetrahedral and the binary octahedral groups as orbifolds in the compactification of the heterotic string is also mentioned [8].

In appendix 1 we give four different decompositions of the root lattice of E_8 under $E_6\times$SU(3) with relevant symmetries indicated. Appendix 2 deals with additional properties of integral octonions. It is shown that there are seven distinct sets of integral octonions each of which equally describe the root lattice of E_8. A simple prescription is suggested for the construction of a desired set.

2. Construction of division algebras with integral elements

According to the celebrated Hurwitz theorem there are only four division algebras formed by real numbers, complex numbers, quaternions and octonions. Let X denote an element over four division algebras. Any element X satisfies a second-order equation (rank equation) with real coefficients

$$X^2-(X+\bar{X})X+X\bar{X}=0 \tag{2.1}$$

where $X+\bar{X}$ is twice the scalar part and $\bar{X}X=X\bar{X}$ is the norm of X. $\bar{X}$ is complex, quaternionic and octonionic conjugation. The set of elements A satisfying (2.1) with $X+\bar{X}=\text{integer}$ and $X\bar{X}=\text{integer}$ are called integral elements over four division algebras provided they obey the following conditions:

(i) A is closed under subtraction and multiplication;

(ii) A contains 1; (2.2)

(iii) A is not a subset of a larger set satisfying (i) and (ii).

It is obvious that ordinary integers m and Gaussian integers $m+in$ (m and n ordinary integers) satisfy (2.2). Equation (2.2) is also satisfied by 24 integral quaternions [9] and 240 integral octonions [4]. In § 3 we will construct integral elements over division algebras starting with ± 1 and show their relations with the root systems of SU(2), SO(4), SO(8) and E_8. For a simple construction we follow the procedure described below.

We may define complex numbers by taking a pair of real numbers. Similarly quaternions and octonions can be defined as pairs of complex numbers and pairs of quaternions respectively. Let P and Q be elements of a given division algebra besides octonions. There does not exist any division algebra beyond octonions. We denote $[P, Q]$ as a pair of such elements and define the product of such pairs by the Cayley-Dixon procedure [10]

$$[P, Q][R, S]=[PR-S\bar{Q}, RQ+\bar{P}S]. \tag{2.3}$$

$[P, Q]$ can also be represented as the sum of these elements where Q is multiplied by an imaginary number on the left, i.e. $[P, Q]=P+eQ$ where $e^2=-1$. The conjugate of such a pair is defined by

$$\overline{[P, Q]}=[\bar{P}, -Q] \tag{2.4}$$

so that the norm of $[P, Q]$ is given as

$$[P, Q]\overline{[P, Q]}=\overline{[P, Q]}[P, Q]=[P\bar{P}+Q\bar{Q}, 0]=P\bar{P}+Q\bar{Q}. \tag{2.5}$$

Here $P\bar{P}$ and $Q\bar{Q}$ are real numbers. Since this description is independent of the choice of the imaginary units, particularly in the case of octonions, we will often use (2.3) for the multiplication of two elements. However we will also define the imaginary units to make contact with other approaches. We define the scalar product of two elements $[P, Q]$ and $[R, S]$ by

$$\begin{aligned}\langle [P, Q], [R, S]\rangle &= \tfrac{1}{2}(\overline{[P, Q]}[R, S]+\overline{[R, S]}[P, Q])\\ &= \tfrac{1}{2}(\bar{P}R+\bar{R}P+Q\bar{S}+S\bar{Q}).\end{aligned} \tag{2.6}$$

Let us now find the relation between this description and those with explicit imaginary units and give examples. Let a_0 and a_1 be two real numbers; then $[a_0, a_1]$ is a complex number which can be written as

$$[a_0, a_1] = a_0 + e_1 a_1 = C \tag{2.7}$$

where $e_1^2 = -1$, $\overline{e_1} = -e_1$. e_1 is used in place of the usual imaginary number i with the idea that we will introduce the imaginary units for quaternions and octonions. Let us take two complex numbers $C = a_0 + e_1 a_1$ and $C' = a_3 + e_1 a_2$ where a_0, a_1, a_2 and a_3 are real numbers. By combining two complex numbers C and C' in the usual manner we obtain a quaternion q given by

$$Q = [C, C'] = a_0 + e_1 a_1 + e_3(a_3 + e_1 a_2) = a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3. \tag{2.8}$$

At this stage we have introduced a second imaginary unit $e_3^2 = -1$, $\overline{e_3} = -e_3$ and defined $e_2 = e_3 e_1 = -e_1 e_3$. Similarly we take a pair of quaternions Q and Q' and obtain an octonion $[Q, Q']$ by multiplying Q' by an imaginary unit e_7 on the left:

$$[Q, Q'] = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + e_7(a_4 e_1 + a_5 e_2 + a_6 e_3 + a_7) = \sum_{i=0}^{7} a_i e_i \tag{2.9}$$

with the definition $e_4 = e_7 e_1 = -e_1 e_7$, $e_5 = e_7 e_2 = -e_2 e_7$, $e_6 = e_7 e_3 = -e_3 e_7$. This is the notation of Gunaydin and Gursey [11] and further properties can be found in their pioneering work. Non-commutativity of quaternions and non-associativity of octonions can be proved by using (2.3). Although we do not need a multiplication formula other than (2.3) the reader may find it helpful to use the multiplication table of imaginary units of octonions defined in (2.9). When three imaginary units of octonions e_a, e_b and e_c satisfy the relation $e_a e_b e_c = -1$ $(a \neq b \neq c)$ we call them associative triads and choose units such that they are represented by seven sets of ordered numbers 123, 246, 435, 367, 651, 572 and 714. There are 28 anti-associative triads which can be obtained using the relations for associative triads.

3. SO(8) root lattice and the binary tetrahedral group

In what follows all integral elements and 'half-integral' elements have norms of unity and $\frac{1}{2}$ respectively. The way we define half-integral element will soon be clear but we should notice that half-integral elements do not satisfy (2.1) and (2.2). Integral and half-integral elements correspond to long and short roots of the relevant groups to be discussed provided they are multiplied by $\sqrt{2}$.

The units of ordinary integers ± 1 are the non-zero roots of SU(2) and its two-dimensional spinor representation is represented by the weights $\pm\frac{1}{2}$. The spinor weights of SU(2) will be essential to obtain the integral quaternions, spinor representations of SO(8) and integral octonions. We start with two sets of ± 1, $\pm\frac{1}{2}$ to obtain integral and half-integral complex numbers providing norms are less than or equal to 1. The following pairs are the only possibilities:

$$[\pm 1, 0] = \pm 1 \qquad [0, \pm 1] = \pm e_1 \tag{3.1a}$$

$$[\pm\tfrac{1}{2}, \pm\tfrac{1}{2}] = \tfrac{1}{2}(\pm 1 \pm e_1). \tag{3.1b}$$

In Lie algebras long and short roots differ by $\sqrt{2}$ besides those of G_2. For this we have not included those terms like $[\pm\frac{1}{2}, 0]$, $[0, \pm\frac{1}{2}]$, $[\pm 1, \pm\frac{1}{2}]$ and $[\pm\frac{1}{2}, \pm 1]$ as they do not correspond to any root at all. Integral complex numbers (Gaussian integers) in (3.1*a*) are the non-zero roots of $SO(4) \approx SU(2) \times SU(2)$ as ± 1 and $\pm e_1$ are orthogonal to each other. Four half-integral complex numbers in (3.1*b*) are weights of the vector representation $(\mathbf{2}, \mathbf{2})$ of SO(4) and together with ± 1 and $\pm e_1$ they represent the non-zero roots of SP(2). This follows from the Coxeter-Dynkin diagram of SP(2) (figure 1) with the simple roots represented by e_1 and $\frac{1}{2}(1-e_1)$. We notice that the roots ± 1 form the discrete Abelian group Z_2 leaving the representations of SU(2) invariant. Similarly ± 1, $\pm e_1$ form the discrete Abelian group Z_4 which is the automorphism group of the root lattice of SO(4). Z_4 also leaves the root system of SP(2) invariant. If we multiply the short roots (3.1*b*) of SP(2) by $\sqrt{2}$ we see that the eight elements of (3.1*a*, *b*) form a larger group Z_8. A similar situation will arise when we discuss the roots of F_4.

Now we come to the crucial point. Although the roots of SP(2) do not generate an integral lattice, when two sets of SP(2) roots are paired as in (2.8) they generate the root lattice of SO(8). To make this clear let us denote the SP(2) roots by set C: ± 1, $\pm e_1$, $\frac{1}{2}(\pm 1 \pm e_1)$. Take two such sets and form pairs of them such that the norms of the pairs will be equal to 1. There are only three possibilities:

$$[(\pm 1, \pm e_1), 0] = (\pm 1, \pm e_1) \qquad [0, (\pm 1, \pm e_1)] = (\pm e_3, \pm e_2) \tag{3.2a}$$

$$[\tfrac{1}{2}(\pm 1 \pm e_1), \tfrac{1}{2}(\pm 1 \pm e_1)] = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3). \tag{3.2b}$$

These were obtained by Hurwitz [9] for the first time and they correspond to 24 discrete points on S_3. Since (3.2*a*) is generated by doubling the roots of $SU(2) \times SU(2)$ they represent the four orthogonal roots of $[SU(2)]^4$. Choosing the simple roots -1, e_1, e_2, e_3 corresponding to each SU(2) we may draw an extended Coxeter-Dynkin diagram of SO(8) by joining each simple root to the simple root $\frac{1}{2}(1-e_1-e_2-e_3)$ (figure 2). Deleting any one of the roots -1, e_1, e_2, e_3 one obtains an SO(8) diagram which leads to the set of 24 roots of SO(8) given by (3.2*a*, *b*). We will prefer deleting -1 so that the heighest weight of the adjoint representation is represented by 1. (3.2*a*, *b*) is essentially equivalent to the branching of the adjoint representation of SO(8) under $[SU(2)]^4$:

$$\mathbf{28} = (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}). \tag{3.3}$$

The first four brackets describe the adjoints of SU(2) each of which has one zero root. To find the connections with the orthogonal vectors u_i ($i = 1, 2, 3, 4$), which is the usual

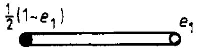

Figure 1. Coxeter-Dynkin diagram of SP(2) with simple roots $\frac{1}{2}(1+e_1)$ and e_1.

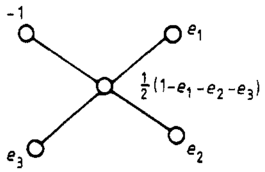

Figure 2. Extended Coxeter-Dynkin diagram of SO(8) (four disconnected roots of $SU(2)^4$ $[(-1, e_1), 0]$ and $[0, (1, e_1)]$ are connected to the root $-[\frac{1}{2}(-1+e_1), \frac{1}{2}(1+e_1)]$).

notation, we define the simple roots as

$$\begin{array}{llll} e_1 = u_1 - u_2 & \frac{1}{2}(1 - e_1 - e_2 - e_3) = u_2 - u_3 & e_2 = u_3 - u_4 & e_3 = u_3 + u_4 \\ u_1 = \frac{1}{2}(1 + e_1) & u_2 = \frac{1}{2}(1 - e_1) & u_3 = \frac{1}{2}(e_3 + e_2) & u_4 = \frac{1}{2}(e_3 - e_2). \end{array} \tag{3.4}$$

One can show that the integral quaternions consist of all expressions $\Sigma_{i=1}^{4} x_i u_i$ where the x_i are integers and $\Sigma_{i=1}^{4} x_i$ is even.

One can easily show that the integral quaternions in (3.2*a*, *b*) satisfy (2.1) and (2.2). Thus 24 integral quaternions form a group called binary tetrahedral group denoted by $\langle 3, 3, 2\rangle$, meaning that it is generated by two independent generators satisfying $S^3 = T^3 = (ST)^2 = Z$, $Z^2 = 1$ [12]. We may represent the quaternionic units by 2×2 matrices as

$$e_0 = \mathbf{I} \qquad e_1 = \mathrm{i}\sigma_2 \qquad e_2 = \mathrm{i}\sigma_1 \qquad e_3 = \mathrm{i}\sigma_3 \tag{3.5}$$

where $\mathbf{I}$ is a 2×2 unit matrix and σ_i $(i = 1, 2, 3)$ are the Pauli matrices. Then we can give the 2×2 matrix representation of (3.2*a*, *b*) which are unitary and of determinant 1. Therefore 24 roots of SO(8) represented by unitary matrices form a discrete subgroup of SU(2). Taking the simple roots as in (3.4) we can show that the following integral elements are positive roots of SO(8):

$$\begin{array}{lll} 1,\ e_1,\ e_2,\ e_3 & S = \frac{1}{2}(1 + e_1 + e_2 + e_3) & T = \frac{1}{2}(1 - e_1 + e_2 + e_3) \\ U = \frac{1}{2}(1 + e_1 - e_2 + e_3) & V = \frac{1}{2}(1 + e_1 + e_2 - e_3) & \bar{S},\ \bar{T},\ \bar{U},\ \bar{V}. \end{array} \tag{3.6}$$

The table of products for these is given in table 1. We notice that the roots of $[\mathrm{SU}(2)]^4$ ± 1, $\pm e_1$, $\pm e_2$, $\pm e_3$ form a subgroup of the binary tetrahedral group of order 8 called a quaternion group. There are four Abelian subgroups generated by *S*, *T*, *U* and *V*, two of which are independent and satisfy

$$\begin{gathered} S^3 = T^3 = U^3 = V^3 = -1 \\ S\bar{S} = T\bar{T} = U\bar{T} = V\bar{V} = 1 \\ S + \bar{S} = T + \bar{T} = U + \bar{U} = V + \bar{V} = 1. \end{gathered} \tag{3.7}$$

The action of a group element *P* on a root *Q* of SO(8) can be defined as

$$Q \to Q' = PQP^{-1} = PQ\bar{P} \tag{3.8}$$

Table 1. Table of products of the elements of the binary tetrahedral group.

	1	e_1	e_2	e_3	S	T	U	V	$\bar{S}$	$\bar{T}$	$\bar{U}$	$\bar{V}$
1	1	e_1	e_2	e_3	S	T	U	V	$\bar{S}$	$\bar{T}$	$\bar{U}$	$\bar{V}$
e_1	e_1	-1	e_3	$-e_2$	$-\bar{U}$	U	$-T$	$-\bar{S}$	V	$-\bar{V}$	S	$\bar{T}$
e_2	e_2	$-e_3$	-1	e_1	$-\bar{V}$	$-\bar{S}$	V	$-U$	T	$\bar{U}$	$-\bar{T}$	S
e_3	e_3	e_2	$-e_1$	-1	$-\bar{T}$	$-V$	$-\bar{S}$	T	U	S	$\bar{V}$	$-\bar{U}$
S	S	$-\bar{V}$	$-\bar{T}$	$-\bar{U}$	$-\bar{S}$	e_3	e_1	e_2	1	V	T	U
T	T	V	$-\bar{U}$	$-\bar{S}$	e_2	$-\bar{T}$	S	$\bar{U}$	$\bar{V}$	1	$-e_1$	e_3
U	U	$-\bar{S}$	T	$-V$	e_3	$\bar{V}$	$-\bar{U}$	S	$\bar{T}$	e_1	1	$-e_2$
V	V	$-T$	$-\bar{S}$	U	e_1	S	$\bar{T}$	$-\bar{V}$	$\bar{U}$	$-e_3$	e_2	1
$\bar{S}$	$\bar{S}$	U	V	T	1	$\bar{U}$	$\bar{V}$	$\bar{T}$	$-S$	$-e_2$	$-e_3$	$-e_1$
$\bar{T}$	$\bar{T}$	$-\bar{U}$	S	$\bar{V}$	U	1	$-e_2$	e_1	$-e_3$	$-T$	V	$\bar{S}$
$\bar{U}$	$\bar{U}$	$\bar{T}$	$-\bar{V}$	S	V	e_2	1	$-e_3$	$-e_1$	$\bar{S}$	$-U$	T
$\bar{V}$	$\bar{V}$	S	$\bar{U}$	$-\bar{T}$	T	$-e_1$	e_3	1	$-e_2$	U	$\bar{S}$	$-V$

which corresponds to a rotation of angle π or $2\pi/3$ in a certain hyperplane depending on the choice of the group element P. Actually any quaternion can be written in the exponential form

$$P=e^{\alpha\Omega}=P_0e_0+P_1e_1+P_2e_2+P_3e_3=\cos\alpha+\Omega\sin\alpha \qquad P\bar{P}=1 \quad (3.9)$$

where $\cos\alpha=\frac{1}{2}(P+\bar{P})$ and $\Omega=(P_1e_1+P_2e_2+P_3e_3)/\sqrt{P_1^2+P_2^2+P_3^2}$ with $\Omega^2=-1$, $\bar{\Omega}=-\Omega$. Therefore a transformation of the form

$$Q\to Q'=e^{\alpha\Omega}Q\,e^{-\alpha\Omega} \quad (3.10)$$

is a rotation of angle 2α in the plane determined by the origin 0, unit vector 1 and P. It is obvious from (3.9) that transformations by $\pm e_i$ $(i=1,2,3)$ and $\frac{1}{2}(\pm1\pm e_1\pm e_2\pm e_3)$ correspond to angles π and $2\pi/3$ respectively.

It is clear from (3.8) that the scalar part of a quaternion is left invariant. Moreover the angle between any two integral quaternion is also left invariant. Therefore the binary tetrahedral group constitutes a discrete subgroup of SU(2), the automorphism group of quaternions, which also preserve the root structure of SO(8). Since the scalar part of a quaternion is left invariant under the binary tetrahedral group it is sufficient to know how the imaginary units e_1, e_2, and e_3 transform under the action of group elements. Using table 1 one can show that under the action of the elements e_1, e_2, e_3, S, T, U and V the imaginary units transform as

$$\pm e_1:\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}\to\begin{bmatrix}e_1\\-e_2\\-e_3\end{bmatrix}\qquad \pm e_2:\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}\to\begin{bmatrix}-e_1\\e_2\\-e_3\end{bmatrix}\qquad \pm e_3:\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}\to\begin{bmatrix}-e_1\\-e_2\\e_3\end{bmatrix} \quad (3.11a)$$

$$\pm S:\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}\to\begin{bmatrix}e_2\\e_3\\e_1\end{bmatrix}\qquad \pm T:\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}\to\begin{bmatrix}-e_3\\-e_1\\e_2\end{bmatrix}\qquad \pm U:\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}\to\begin{bmatrix}e_3\\-e_1\\-e_2\end{bmatrix}$$

$$\pm V:\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix}\to\begin{bmatrix}-e_3\\e_1\\-e_2\end{bmatrix}. \quad (3.11b)$$

It is clear from (3.11a, b) that the quaternion algebra is left invariant under the binary tetrahedral group. The group element S rotates the simple roots e_1, e_2 and e_3 in cyclic order. On the other hand we know that the outer automorphism of the SO(8) root lattice is the symmetric group S_3 generated by an element which permutes the simple roots e_1, e_2 and e_3 in cyclic order and an element A which exchanges the order of two roots, say, $e_2\Leftrightarrow e_3$. The element S of S_3 is nothing other than the element S of the binary tetrahedral group. However A is not in the set of elements of the binary tetrahedral group as it does not preserve the quaternion algebra. We shall see later that S, T, U and V play important roles concerning the decomposition of E_8 roots with respect to its maximal subgroup $E_6\times SU(3)$. S preserves the level of the SO(8) roots which has been used to determine the cocyles arising in the vertex operator construction of affine F_4 algebra [7].

Now we calculate the weights of three eight-dimensional representations of SO(8) in terms of quaternionic units. We adopt the usual notation $8_v=(1\ 0\ 0\ 0)$, $8_s=(0\ 0\ 0\ 1)$ and $8_c=(0\ 0\ 1\ 0)$ for the vector, spinor and antispinor representations respectively in the Dynkin formalism [13]. It is straightforward to obtain the following weights for three eight-dimensional representations of SO(8) as

$$\begin{array}{llllll} & \underline{\quad A_1 \quad} & & \underline{\quad A_3 \quad} & & \underline{\quad A_2 \quad} \\ 8_v: & \tfrac{1}{2}(\pm 1 \pm e_1) & 8_s: & \tfrac{1}{2}(\pm 1 \pm e_3) & 8_c: & \tfrac{1}{2}(\pm 1 \pm e_2) \\ & \tfrac{1}{2}(\pm e_2 \pm e_3) & & \tfrac{1}{2}(\pm e_1 \pm e_2) & & \tfrac{1}{2}(\pm e_3 \pm e_1). \end{array} \tag{3.12}$$

Since the choices for the representations v, s, c are arbitrary we have also denoted them by other labels A_1, A_3 and A_2 denoting the whole set of corresponding eight weights. We note that each imaginary unit characterises one of the eight-dimensional representations. In fact the highest weight of the adjoint representation of SO(8) equals 1. Thus the conjugacy classes of SO(8) representations are represented by four units of quaternions 1, e_1, e_2 and e_3. Using (3.4) one can also express (3.12) in terms of the orthogonal vectors u_i $(i=1,2,3,4)$. We also note that the group elements $\bar{T}$, $\bar{U}$ and $\bar{V}$ rotate the eight-dimensional representations in the cyclic order $8_v \to 8_s \to 8_c \to 8_v$. In contrast, S, $\bar{T}$, $\bar{U}$ and $\bar{V}$ do not preserve the levels of the weights. On the other hand the quaternion subgroup leaves each representation invariant while changing the levels of the weights within a given representation. These properties will be fully utilised in the construction of the E_8 root lattice.

Similar to the case in SP(2), if we multiply the weights in (3.12) by $\sqrt{2}$ and combine them with the 24 integral quaternions (3.2a, b) we obtain a new discrete subgroup of SU(2) of order 48 which admits the binary tetrahedral group as a subgroup. This new group, denoted by $\langle 4, 3, 2\rangle$, is called a binary octahedral group [12]. We will show in the next section that the 24 integral quaternions (3.2a, b) and the 24 weights in (3.12) form the root system of F_4 with 24 long and 24 short roots respectively.

4. Integral octonions and their symmetries

This section and § 5 will constitute the main parts of our work as they are related to the root lattice of E_8. First we notice that the SO(8) root lattice with its weights of vector representation 8_v in (3.12) are the roots of SO(9) where the remaining two spinor representations in (3.12) constitute the 16-dimensional spinor representation of SO(9). In pairing two SP(2) roots, if we had allowed pairings where norms less than 1 were allowed, we would have uniquely obtained the SO(9) root system. F_4 admits SO(9) as the maximal subgroup so that the 48 non-zero roots of F_4 decomposes under SO(9) as

$$48 = 32 + 16. \tag{4.1}$$

Here 32 is the number of non-zero roots of SO(9). Equation (4.1) further splits under SO(8) as

$$48 = 24 + 8_v + 8_s + 8_c. \tag{4.2}$$

We follow the same procedure described in § 3 and first combine two SO(9) roots and check whether they correspond to any familiar root lattice. In fact when we pair two sets of SO(9) roots of A_0 and A_1 we see that there is a unique way of obtaining a set of 112 integral octonions of unit norm which constitute the root lattice of SO(16). This pairing can be arranged as follows:

$$112 = [A_0, 0] + [0, A_0] + [A_1, A_1]. \tag{4.3}$$

This is in accord with the branching of SO(16) with respect to its maximal subgroup SO(8)×SO(8):

$$\mathbf{120} = (\mathbf{28}, \mathbf{1}) + (\mathbf{1}, \mathbf{28}) + (\mathbf{8}_v, \mathbf{8}_v) \tag{4.4}$$

An explicit representation of 112 non-zero roots of SO(16) can be written in terms of integral octonions as follows:

$$\begin{aligned}
&[A_0, 0]: && \pm 1, \quad \pm e_1, \quad \pm e_2, \quad \pm e_3, \quad \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \\
&[0, A_0]: && \pm e_7, \quad \pm e_4, \quad \pm e_5, \quad \pm e_6, \quad \tfrac{1}{2}(\pm e_4 \pm e_5 \pm e_6 \pm e_7) \\
&[A_1, A_1]: && [\tfrac{1}{2}(\pm 1 \pm e_1), \tfrac{1}{2}(\pm 1 \pm e_1)] = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_4 \pm e_7) \\
& && [\tfrac{1}{2}(\pm 1 \pm e_1), \tfrac{1}{2}(\pm e_2 \pm e_3)] = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_5 \pm e_6) \\
& && [\tfrac{1}{2}(\pm e_2 \pm e_3), \tfrac{1}{2}(\pm 1 \pm e_1)] = \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_4 \pm e_7) \\
& && [\tfrac{1}{2}(\pm e_2 \pm e_3), \tfrac{1}{2}(\pm e_2 \pm e_3)] = \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6).
\end{aligned} \tag{4.5}$$

We shall see that this set is not closed under multiplication. The next pairing must be done among the roots of F_4 in (4.2) which combines three eight-dimensional representations of length 1 and the roots of adjoint of SO(8) of length square 2. It is interesting to note that the long and short roots of F_4 split in equal numbers of 24 each. A similar situation occurred in the case of SP(2). The simple roots of F_4 (figure 3) can be chosen as

$$\alpha_1 = \tfrac{1}{2}(1 - e_1 - e_2 - e_3) \qquad \alpha_2 = e_2 \qquad \alpha_3 = \tfrac{1}{2}(e_3 - e_2) \qquad \alpha_4 = \tfrac{1}{2}(e_1 - e_3). \tag{4.6}$$

When we calculate the roots of F_4 with this choice of simple roots we exactly get a collection of A_0, A_1, A_2 and A_3 where the positive roots are given by

$$1, \quad e_1, \quad e_2, \quad e_3 \quad \tfrac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3) \tag{4.7a}$$

$$\tfrac{1}{2}(1 \pm e_1) \qquad \tfrac{1}{2}(\pm e_2 + e_3) \tag{4.7b}$$

$$\tfrac{1}{2}(1 \pm e_3) \qquad \tfrac{1}{2}(\pm e_1 + e_2) \tag{4.7c}$$

$$\tfrac{1}{2}(1 \pm e_2) \qquad \tfrac{1}{2}(\pm e_3 + e_1). \tag{4.7d}$$

Here (4.7*a*, *b*, *c*, *d*) denote the positive weights of the adjoint, 8_v, 8_s and 8_c of SO(8) respectively. Here again the quaternionic unit 1 represents the heighest root of F_4 as in the case of SO(8). The weights in (3.12) also constitute the 24 non-zero weights of the 26-dimensional representation of F_4.

After this brief remark we are ready to construct the root lattice of E_8 with integral octonions. Since we have already constructed the root lattice of the SO(16) subgroup

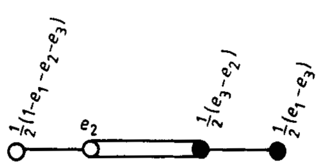

Figure 3. Coxeter-Dynkin diagram of F_4 with integral and half-integral quaternions.

of E_8, what remains is a pairing of two sets of A_2+A_3. At this stage an ambiguity arises. We may have the following pairings:

$$[A_2, A_2] \qquad [A_3, A_3] \tag{4.8a}$$

$$[A_2, A_3] \qquad [A_3, A_2] \tag{4.8b}$$

for the 128-dimensional spinor representation of SO(16) to be added to 112 non-zero roots to yield 240 non-zero roots of E_8. Which of them would constitute the right structure can be tested by the conditions stated in (2.2). The crucial criterion here is the closure of the set under the octonionic multiplication. We will show that the true structure will emerge only if we take the pairs in (4.8*b*). To prove this we calculate the products of the integral octonions given by (4.3) and show that their products generate two additional pairs $[A_2, A_3]$ and $[A_3, A_2]$. Let us set up the required products by using (2.3):

$$[A_0, 0][A_0', 0] = [A_0A_0', 0] \subset [A_0, 0] \tag{4.9a}$$

$$[A_0, 0][0, A_0'] = [0, \bar{A}_0A_0'] \subset [0, A_0] \tag{4.9b}$$

$$[0, A_0][0, A_0'] = [-A_0'\bar{A}_0, 0] \subset [A_0, 0] \tag{4.9c}$$

$$[A_0, 0][A_1, A_1'] = [A_0A_1, \bar{A}_0A_1'] \subset [M, N] \tag{4.9d}$$

$$[0, A_0][A_1, A_1'] = [-A_1'\bar{A}_0, A_1A_0] \subset [M, N] \tag{4.9e}$$

$$[A_1, A_1'][B_1, B_1'] = [A_1B_1 - B_1'\bar{A}_1', B_1A_1' + \bar{A}_1B_1'] \subset [C, D]. \tag{4.9f}$$

Equations (4.9*a*, *b*) mean that the set of integral octonions $[A_0, 0]$ and $[0, A_0]$ preserve their form by a left product of quaternions. $[0, A_0]$ represents a special set of integral octonions as they are obtained by multiplying the integral quaternions by e_7 on the left. The product of this special set with itself yields 24 integral quaternions $[A_0, 0]$ presented in (4.9*c*). Equations (4.9*d*, *e*, *f*) deserve more attention as their structures are more involved. Before analysing them a few remarks are necessary about the products of the form A_0A_1 and $\bar{A}_0A_1$. The integral quaternions of A_0 can be split into two sets of elements, one set results in $A_0A_1 \to A_1$ and $\bar{A}_0A_1 \to A_1$; the other set of elements yields products of the form $A_0A_1 = A_3$ and $\bar{A}_0A_1 = A_2$ and vice versa. To give a simple example let us take e_1 from A_0 and $\frac{1}{2}(1+e_1)$ from A_1. It is clear that a left multiplication of $\frac{1}{2}(1+e_1)$ by e_1 and $\bar{e}_1$ leaves $\frac{1}{2}(1+e_1)$ in the same set A_1. Indeed not only e_1 but the whole elements ± 1, $\pm e_1$, $\pm e_2$, $\pm e_3$ of the quaternion group satisfy this property. The remaining elements S, T, U, V and their conjugates with their negatives satisfy the second condition. To make this point also clear let us choose S from A_0 and $\frac{1}{2}(e_3+e_2)$ and $\frac{1}{2}(1+e_1)$ from A_1. Using table 1 we can write the products in (4.9*d*) as

$$S\tfrac{1}{2}(e_3+e_2) = -\tfrac{1}{2}(\bar{U}+\bar{T}) = \tfrac{1}{2}(-1+e_3) \tag{4.10a}$$

$$\bar{S}\tfrac{1}{2}(1+e_1) = \tfrac{1}{2}(\bar{S}+U) = \tfrac{1}{2}(1-e_2). \tag{4.10b}$$

Thus the above statement is true as $\frac{1}{2}(-1+e_3)$ and $\frac{1}{2}(1-e_2)$ belong to the sets A_3 and A_2 respectively. To see the validity of our statement for all elements of A_0 and A_1 we recall (3.11*b*) that the elements S, T, U and V always map a given set of weights to another set of weights so that the corresponding maps of $\bar{S}$, $\bar{T}$, $\bar{U}$ and $\bar{V}$ do not coincide. Hence the right-hand side of (4.9*d*) should be either $[A_2, A_3]$ or $[A_3, A_2]$. For the same reason (4.9*e*) should be also in the same form, i.e. $[M, N]$ should be either $[A_2, A_3]$ or $[A_3, A_2]$. The analysis of (4.9*f*) requires more effort; nevertheless one can show that the general structure of $[C, D]$ is one of the forms $[A_0, 0]$, $[0, A_0]$, $[A_1, A_1]$, $[A_2, A_3]$ or $[A_3, A_2]$ but nothing else. To summarise, we can generate the root lattice of E_8 in the form of integral octonions from the root lattice of SO(16) by performing multiple products of the elements of the set (4.5). When the terms $[A_2, A_3]$ and $[A_3, A_2]$ are added to (4.4) or (4.5) we get 240 integral octonions representing the root lattice of E_8. To check whether the new set is closed under multiplication we calculate the remaining products and show that the new set is really closed. To give a counter-example that two types of terms of the forms $[A_1, A_1]$ and $[A_2, A_2]$ cannot be in a given set of integral octonions representing the roots of E_8 let us consider the following product of these pairs:

$$[A_1, A_1][A_2, A_2]: [\tfrac{1}{2}(1+e_1), \tfrac{1}{2}(e_2+e_3)][\tfrac{1}{2}(1+e_2), \tfrac{1}{2}(1+e_2)] = [\tfrac{1}{2}(e_1+e_2+e_3), \tfrac{1}{2}e_2]. \tag{4.11}$$

Since the terms in the bracket on the right-hand side of (4.11) do not correspond to eight-dimensional representations of SO(8) this term cannot be in the required set of integral octonions of E_8. Therefore the set of the integral octonions satisfying (2.1) and (2.2) and representing the root lattice of E_8 must be of the form

$$240 = [A_0, 0] + [0, A_0] + [A_1, A_1] + [A_3, A_2] + [A_2, A_3] \tag{4.12}$$

This represents an interesting pairing of two sets of the 48 roots of F_4: $A_0 + A_1 + A_2 + A_3$.

Let us now investigate the problem from a different point of view. We wish to discuss how two Coxeter–Dynkin diagrams of SO(8) can be connected to construct a Coxeter–Dynkin diagram of E_8. We take two sets of Coxeter–Dynkin diagrams of SO(8) whose simple roots are expressed as in (3.4) and multiply the quaternionic simple roots in one set by e_7 on the left and then connect them to form an extended diagram of SO(16) by adding a simple root linking two simple roots, one from each SO(8) (figure 4). To end up with an extended Coxeter–Dynkin diagram of E_8 we delete one of the roots in figure 4 and add one more root to right end of the diagram figure 5. At this point another puzzling situation arises. Namely, if we keep the orders of simple roots in two SO(8) diagrams we always generate two sets of roots in the form $[A_i, A_i]$ and $[A_j, A_j]$ $(i \neq j = 1, 2, 3)$. Since we have already proved that two sets like these terms should be out of the root lattice of E_8 the combination of two SO(8) diagrams in this manner is not allowed. However there is a simple solution: one can interchange the orders of two simple roots say $e_2 \Leftrightarrow e_3$ in one of the diagram and then combine. Then one obtains a legal extended Coxeter–Dynkin diagram of E_8 (figure 5). When we change the orders of two simple roots in one of the SO(8) diagrams we

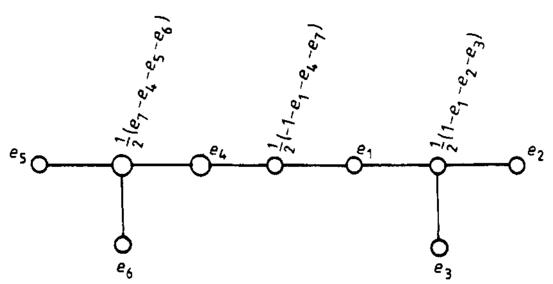

Figure 4. Extended Coxeter–Dynkin diagram of SO(16) (the diagram is made by combining two SO(8) diagrams in one of which two roots $e_2 \Leftrightarrow e_3$ is interchanged and the roots of this SO(8) is multiplied by e_7 on the left).

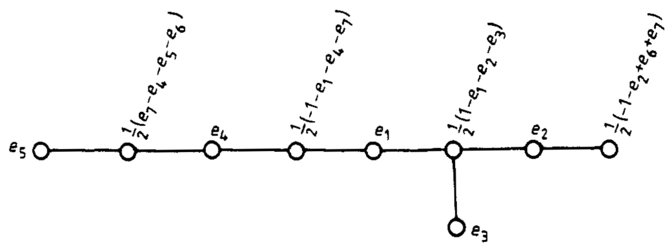

Figure 5. Extended Coxeter-Dynkin diagram of E_8 (integral quaternions representing the SO(8) subgroup are manifest).

also change the label of two spinor representations. While in the first diagram, say, spinor and antispinor are represented by the set of weights A_3 and A_2 respectively; in the second the labels of the representations are reversed.

Thus we obtain a decomposition of E_8 under SO(8) × SO(8)

$$\mathbf{248} = (\mathbf{28}, \mathbf{1}) + (\mathbf{1}, \mathbf{28}) + (\mathbf{8}_v, \mathbf{8}_v) + (\mathbf{8}_s, \mathbf{8}_s) + (\mathbf{8}_c, \mathbf{8}_c). \tag{4.13}$$

The non-zero roots of E_8 are represented by the following integral octonions:

$$(24, 1): \quad \pm 1, \quad \pm e_1, \quad \pm e_2, \quad \pm e_3, \quad \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \tag{4.14a}$$

$$(1, 24): \quad \pm e_7, \quad \pm e_4, \quad \pm e_5, \quad \pm e_6, \quad \tfrac{1}{2}(\pm e_7 \pm e_4 \pm e_5 \pm e_6) \tag{4.14b}$$

$$\begin{aligned}(\mathbf{8}_v, \mathbf{8}_v): \quad &[\tfrac{1}{2}(\pm 1 \pm e_1), \tfrac{1}{2}(\pm 1 \pm e_1)] = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_4 \pm e_7)\\ &[\tfrac{1}{2}(\pm 1 \pm e_1), \tfrac{1}{2}(\pm e_3 \pm e_2)] = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_5 \pm e_6)\\ &[\tfrac{1}{2}(\pm e_3 \pm e_2), \tfrac{1}{2}(\pm 1 \pm e_1)] = \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_4 \pm e_7)\\ &[\tfrac{1}{2}(\pm e_3 \pm e_2), \tfrac{1}{2}(\pm e_3 \pm e_2)] = \tfrac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6)\end{aligned} \tag{4.14c}$$

$$\begin{aligned}(\mathbf{8}_s, \mathbf{8}_s): \quad &[\tfrac{1}{2}(\pm 1 \pm e_3), \tfrac{1}{2}(\pm 1 \pm e_2)] = \tfrac{1}{2}(\pm 1 \pm e_3 \pm e_5 \pm e_7)\\ &[\tfrac{1}{2}(\pm 1 \pm e_3), \tfrac{1}{2}(\pm e_3 \pm e_1)] = \tfrac{1}{2}(\pm 1 \pm e_3 \pm e_4 \pm e_6)\\ &[\tfrac{1}{2}(\pm e_1 \pm e_2), \tfrac{1}{2}(\pm 1 \pm e_2)] = \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_5 \pm e_7)\\ &[\tfrac{1}{2}(\pm e_1 \pm e_2), \tfrac{1}{2}(\pm e_3 \pm e_1)] = \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_4 \pm e_6)\end{aligned} \tag{4.14d}$$

$$\begin{aligned}(\mathbf{8}_c, \mathbf{8}_c): \quad &[\tfrac{1}{2}(\pm 1 \pm e_2), \tfrac{1}{2}(\pm 1 \pm e_3)] = \tfrac{1}{2}(\pm 1 \pm e_2 \pm e_6 \pm e_7)\\ &[\tfrac{1}{2}(\pm 1 \pm e_2), \tfrac{1}{2}(\pm e_1 \pm e_2)] = \tfrac{1}{2}(\pm 1 \pm e_2 \pm e_4 \pm e_5)\\ &[\tfrac{1}{2}(\pm e_3 \pm e_1), \tfrac{1}{2}(\pm 1 \pm e_3)] = \tfrac{1}{2}(\pm e_1 \pm e_3 \pm e_6 \pm e_7)\\ &[\tfrac{1}{2}(\pm e_3 \pm e_1), \tfrac{1}{2}(\pm e_1 \pm e_2)] = \tfrac{1}{2}(\pm e_1 \pm e_3 \pm e_4 \pm e_5).\end{aligned} \tag{4.14e}$$

To make contact with the generally accepted notation for simple roots derived from orthogonal vectors u_i $(i = 1, 2, 3, 4)$ of (3.4) we introduce

$$\begin{aligned}l_1 &= [0, -\tfrac{1}{2}(e_3 + e_2)] = [0, -u_3] = -\tfrac{1}{2}(e_5 + e_6)\\ l_2 &= [0, -\tfrac{1}{2}(1 - e_1)] = [0, -u_2] = \tfrac{1}{2}(e_4 - e_7)\\ l_3 &= [0, -\tfrac{1}{2}(1 + e_1)] = [0, -u_1] = -\tfrac{1}{2}(e_4 + e_7)\\ l_4 &= [\tfrac{1}{2}(1 + e_1), 0] = [u_1, 0] = \tfrac{1}{2}(1 + e_1)\\ l_5 &= [\tfrac{1}{2}(1 - e_1), 0] = [u_2, 0] = \tfrac{1}{2}(1 - e_1)\\ l_6 &= [\tfrac{1}{2}(e_3 + e_2), 0] = [u_3, 0] = \tfrac{1}{2}(e_3 + e_2)\\ l_7 &= [\tfrac{1}{2}(e_3 - e_2), 0] = [u_4, 0] = \tfrac{1}{2}(e_3 - e_2)\\ l_8 &= [0, -\tfrac{1}{2}(e_3 - e_2)] = [0, -u_4] = \tfrac{1}{2}(e_5 - e_6).\end{aligned} \tag{4.15}$$

For a simple representation of E_8 in terms of its maximal subgroup SU(9) we also introduce another vector l_0 defined by

$$l_0 = \tfrac{1}{2}\sum_{r=1}^{8} l_r = \tfrac{1}{2}(1+e_3-e_6-e_7) = [\tfrac{1}{2}(1+e_3), -\tfrac{1}{2}(1+e_3)]. \tag{4.16}$$

With these vectors the simple roots in the extended Coxeter–Dynkin diagram of E_8 in figure 5 are represented from left to right by

$$l_8-l_1,\quad l_1-l_2,\quad l_2-l_3,\quad l_3-l_4,\quad l_4-l_5,\quad l_5-l_6,\quad l_6-l_7,\quad l_7-l_0,\quad l_6+l_7. \tag{4.17}$$

Because of triality attributing labels to the eight-dimensional representations of SO(8) is completely arbitrary we could have equally represented the vector representation of SO(8) by the sets A_2 and/or A_3. Then we could have started with $[A_2, A_2]$ or $[A_3, A_3]$ instead of $[A_1, A_1]$ to construct the root lattice of SO(16). Then we would have generated the rest of the roots of E_8 with a similar procedure to that described above. However the new sets of integral octonions would be completely different from those of (4.14). In appendix 2 we show that the integral octonions representing the root lattice of E_8 can be built in seven distinct ways which can be obtained by replacing the imaginary units in (4.14) by the cyclic order $1\to 2\to 3\to 4\to 6\to 5\to 7\to 1$ and by its repeated application. Therefore one should be very careful in distinguishing the integral octonions representing the root lattice of E_8. After all there are $\binom{8}{4}\times 16 = 1120$ distinct integral octonions of the form $\frac{1}{2}(\pm e_a \pm e_b \pm e_c \pm e_d)$ $(a \neq b \neq c \neq d \neq = 0, 1, \ldots, 7)$ which are distributed in 7 sets of 240 integral octonions so that each set separately satisfies (2.1) and (2.2).

Before we end this section we discuss the action of the binary tetrahedral group on the rest of integral octonions in (4.14). Let P be an element of the binary tetrahedral group, i.e. an integral quaternion of $[A_0, 0]$. Let R represents an octonion of the form $[M, N]$, not necessarily of unit norm. We can define the action of P on R in the form similar to the one given by (3.8)

$$R \to R' = PR\bar{P} \qquad \bar{P} = P^{-1}. \tag{4.18}$$

To have this make sense the right-hand side of (4.18) must be associative. Indeed, this is the case. To prove this we can calculate the associator of the elements $[P, 0]$, $[M, N]$, $[Q, 0]$:

$$([P,0],[M,N],[Q,0]) = [PMQ - PMQ,\ Q\bar{P}N - \bar{P}QN]. \tag{4.19}$$

Since P, M, N and Q are quaternions each triple product in the bracket is associative. To make (4.19) vanish we should have $Q\bar{P} = \bar{P}Q$. This is satisfied for $Q = \bar{P}$ and $Q = P$. Therefore (4.18) is an unambiguous transformation of the form

$$[P,0][M,N][\bar{P},0] = [PM\bar{P}, \bar{P}^2 N] \tag{4.20}$$

provided we take $Q = \bar{P}$. If we had taken $Q = P$ this would also correspond to a rotation in a hyperplane perpendicular to the hyperplane defined by 0, 1 and P. Now we check how the integral octonions (4.14) transform under the action of the elements of the binary tetrahedral group. Since we have already seen the transformations of the integral quaternions of the form $[A_0, 0]$ we investigate the transformations of the remaining terms in the forms $[0, A_0]$ and $[M, N]$ in (4.14). It is clear from (4.20) that the $[0, A_0]$ preserves its form:

$$P\colon\quad [0, A_0] \to [0, \bar{P}^2 A_0] = [0, A_0]. \tag{4.21}$$

By using (3.11a, b) and (4.10a, b) we can show that the right-hand side of (4.20) remains always in the form $[A_1, A_1]$, $[A_3, A_2]$ and $[A_2, A_3]$. To become more explicit let us choose the elements from the quaternion subgroup and the Abelian subgroups of S, T, U and V. Let $[P, 0]$ be an element of quaternion subgroup of the binary tetrahedral group. By (3.11a) we can show that the left-hand terms in (4.20) remain in the same set of weights $A_i = PA_i\bar{P}$. The action of the quaternion group elements on the right-hand side leave the element N either unchanged or simply multiply by (-1). Therefore the quaternion group acts as follows:

$$[P, 0][A_i, A_j][\bar{P}, 0] = [A_i, A_j] \qquad (i = 1, 2, 3). \tag{4.22}$$

Thus the quaternion subgroup leaves the terms $[A_0, 0]$, $[0, A_0]$ and $[A_i, A_j]$ separately invariant.

The actions of the Abelian subgroups can be done in a similar way. Let us do it for S. It is clear from (4.20) and (4.21) that the terms $[A_0, 0]$ and $[0, A_0]$ are left separately invariant under S. Let us see how it acts on the other terms. Using (4.20) we write

$$[S, 0][A_1, A_1][\bar{S}, 0] = [SA_1\bar{S}, \bar{S}^2 A_1] = [SA_1\bar{S}, -SA_1]. \tag{4.23a}$$

Using (3.11b) and table 1 we can show that $SA_1\bar{S} = A_2$ and $-SA_1 = A_3$. Therefore (4.23a) is in the form $[A_2, A_3]$. Similarly we can also show that

$$[S, 0][A_2, A_3][\bar{S}, 0] = [A_3, A_2] \tag{4.23b}$$

$$[S, 0][A_3, A_2][\bar{S}, 0] = [A_1, A_1]. \tag{4.23c}$$

Equations (4.23a, b, c) indicate that S rotates the sets $[A_1, A_1]$, $[A_2, A_3]$ and $[A_3, A_2]$ in the cyclic order. The other Abelian subgroups T, U and V leave the set of roots of E_8 (4.14) invariant. Thus we can conclude that the root lattice of E_8 is preserved under the action of the elements of the binary tetrahedral group. We believe that this feature of the E_8 root lattice is useful in the vertex construction of the level one representation of the affine E_8 algebra (for a review, see [14]).

5. Maximal subgroups of E_8 and the construction of the GNORS magic square

In this section we give the branching of 240 roots of E_8 with respect to its subgroups SO(16), $E_7 \times SU(2)$, $E_6 \times SU(3)$, SU(9), $SU(5) \times SU(5)$ and $F_4 \times G_2$ (for a construction of E_8 algebra with respect to its maximal subgroups, see [15]). The decomposition of a simple Lie group with respect to its maximal and regular subgroups can be found in the standard references [16]. We identify the subgroups of the binary tetrahedral group preserving the coset structures of E_8 with respect to its subgroups. We decompose the roots of E_8 in such a way that a magic square similar to that of Freudenthal, Rozenfeld and Tits [17] is transparent. This magic square obtained by GNORS [7] plays an important role in the vertex operator construction for level one representations of the groups of the magic square. Moreover we find their symmetries under the discrete subgroup of SU(2) with the hope that some additional features may arise concerning the Kac–Moody algebras based on these groups.

5.1. SO(16)

SO(16) is one of the maximal subgroup of E_8. The most interesting thing in our construction is not SO(16) but its subgroup $SO(8) \times SO(8)$. We have already given the roots of SO(16) in (4.5) in terms of integral octonions. Therefore the remaining roots $(8_s, 8_s)+(8_c, 8_c)$ of (4.14*d*, *e*) constitute an 128-dimensional spinor representation of SO(16). Since we have already noted that the quaternion group preserves the labels of vector, spinor and antispinor representations, this is reflected as the symmetry of the root lattice of SO(16) and the 128-dimensional spinor representations defining the coset space of $E_8/SO(16)$. We can also give the representations of SO(16) in terms of the vectors defined in (4.15). The roots of E_8 decomposes then as 112 like $\pm l_i \pm l_j$, $i \neq j$ and 128 like $\frac{1}{2}(-l_1+l_2+l_3+l_4+l_5+l_6+l_7+l_8)$ with an odd minus sign in the latter.

5.2. $E_7 \times SU(2)$

This group will be recognised as the maximal subgroup of E_8 when the second root from the left in the extended Coxeter-Dynkin diagram (figure 5) is deleted. Then the diagram splits into two disjoint diagrams where the single disconnected root is describing SU(2) and the remaining diagram is of E_7. The roots in figure 5 are organised in such a way that SO(8) represented by integral quaternions remains always as a subgroup of one of the factor groups besides SU(9) and SU(5)×SU(5). Therefore in the construction of root lattices and the cosets formed by the maximal subgroups we will often use the adjoint, vector and spinor representations of SO(8). In this case SU(2) is represented by $\pm e_5$. Because $E_7 \times SU(2)$ is maximal in E_8 the roots of E_7 must be orthogonal to e_5 (e_5 is represented by e_2 on the right-hand side of the bracket). Therefore the roots of E_7 are such that e_2 on the right-hand sides of the brackets is missing. Therefore the 126 non-zero roots of E_7 are nothing other than

$$\pm 1, \quad \pm e_1, \quad \pm e_2, \quad \pm e_3, \quad \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \tag{5.1a}$$

$$[0, (\pm e_1, \pm e_3, \pm 1)] = \pm e_4, \pm e_6, \pm e_7 \tag{5.1b}$$

$$[A_1, \tfrac{1}{2}(\pm 1 \pm e_1)] \qquad [A_2, \tfrac{1}{2}(\pm 1 \pm e_3)] \qquad [A_3, \tfrac{1}{2}(\pm e_3 \pm e_1)]. \tag{5.1c}$$

Where A_1, A_2, and A_3 are given by (3.12). We recall that (5.1*a*) together with A_1, A_2 and A_3 constitute the roots of F_4. One can show that (5.1*b*) and the half-integral quaternions on the right-hand sides of brackets (5.1*c*) are the roots of SP(3). To check this we use the Dynkin diagram of SP(3) (figure 6) with the simple roots represented by

$$\alpha_1 = \tfrac{1}{2}(-1+e_1) \qquad \alpha_2 = \tfrac{1}{2}(1-e_3) \qquad \alpha_3 = e_3. \tag{5.2}$$

We obtain 18 non-zero roots of SP(3) by Weyl reflections which lead to

$$\pm 1, \quad \pm e_1, \quad \pm e_3, \quad \tfrac{1}{2}(\pm 1 \pm e_1), \quad \tfrac{1}{2}(\pm e_1 \pm e_3), \quad \tfrac{1}{2}(\pm 1 \pm e_3). \tag{5.3}$$

Here the first six roots are the long roots and the rest are the short roots. Thus (5.1*a*, *b*, *c*) can be regarded as a pairing of the roots of F_4 and SP(3) each of which is represented by quaternionic roots. This corresponds to the construction of E_7 by 'matching' the F_4 and SP(3) root system discussed in [5]. In the preceding section we have pointed out that there are several ways of linking two SO(8) diagrams leading

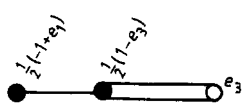

Figure 6. Coxeter-Dynkin diagram of SP(3).

to a proper E_8 diagram which gives the same set of roots of E_8 in (4.14). This arbitrariness is reflected here in the choice of the roots of SU(2) from the quaternionic units ± 1, $\pm e_1$, $\pm e_2$, $\pm e_3$. Since we have already illustrated the case with the selection of e_2 the remaining choices follow the same procedure. Therefore we may choose the roots of SU(2) four different ways and construct the roots of SP(3) accordingly and pair them with the unaltered roots of F_4 in four different ways. This indicates that not only $E_7 \times SU(2)$ can be embedded in E_8 four possible ways but also E_7 can be constructed four different ways by pairing the roots of F_4 and SP(3). After the roots of $E_7 \times SU(2)$ are subtracted from (4.14) the remaining roots transform as a $(\mathbf{56}, \mathbf{2})$-dimensional representation of $E_7 \times SU(2)$. They can be constructed by replacing the half-integral quaternions on the right-hand side of the brackets in (5.1*c*) by the terms $\frac{1}{2}(\pm e_2 \pm e_3)$, $\frac{1}{2}(\pm e_1 \pm e_2)$ and $\frac{1}{2}(\pm 1 \pm e_2)$ respectively. This will give 96 roots only:

$$[A_1, \tfrac{1}{2}(\pm e_2 \pm e_3)] \qquad [A_2, \tfrac{1}{2}(\pm e_1 \pm e_2)] \qquad [A_3, \tfrac{1}{2}(\pm 1 \pm e_2)]. \tag{5.4}$$

When we add to these the 16 terms $\frac{1}{2}(\pm e_7 \pm e_4 \pm e_5 \pm e_6)$ we obtain the right number 112 roots of the coset space of $E_8/E_7 \times SU(2)$. We note also that the root lattice of $E_7 \times SU(2)$ and hence its $(\mathbf{56}, \mathbf{2})$ dimensional representations are left invariant under the quaternion group.

Before discussing other maximal subgroups of E_8 it is perhaps appropriate here to point out that if two sets of SP(3) roots of (5.3) are paired we obtain the root lattice of SO(12). Since (4.14) restricts us we have the only choice for pairing as follows:

$$\begin{aligned}
&\pm 1, \quad \pm e_1, \quad \pm e_3, \quad [0, (\pm 1, \pm e_1, \pm e_3)] = \pm e_7, \pm e_4, \pm e_6 \\
&[\tfrac{1}{2}(\pm 1 \pm e_1), \tfrac{1}{2}(\pm 1 \pm e_1)] = \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_4 \pm e_7) \\
&[\tfrac{1}{2}(\pm 1 \pm e_3), \tfrac{1}{2}(\pm e_1 \pm e_3)] = \tfrac{1}{2}(\pm 1 \pm e_3 \pm e_4 \pm e_6) \\
&[\tfrac{1}{2}(\pm e_1 \pm e_3), \tfrac{1}{2}(\pm 1 \pm e_3)] = \tfrac{1}{2}(\pm e_1 \pm e_3 \pm e_6 \pm e_7).
\end{aligned} \tag{5.5}$$

These are the 60 non-zero roots of SO(12) occurring in the magic square of [7]. Here again we can construct the root lattice of SO(12) four different ways provided we remain in the same set of E_8 roots of (4.14). Equation (5.5) is also invariant under the action of the quaternion group.

We will continue constructing the root systems of the magic square with integral octonions when they become relevant. Now we discuss the branching of the roots of E_8 under $E_6 \times SU(3)$.

5.3. $E_6 \times SU(3)$

If we break up the extended Coxeter-Dynkin diagram in figure 5 by deleting e_4 we obtain the disconnected diagrams for SU(3) and E_6. Using the Weyl reflections formula we obtain the roots of SU(3):

$$\begin{aligned}
&[0, \pm e_2, \pm\tfrac{1}{2}(1 - e_1 - e_2 - e_3), \pm\tfrac{1}{2}(1 - e_1 + e_2 - e_3)] \\
&\qquad = \pm e_5, \tfrac{1}{2}(e_7 - e_4 - e_5 - e_6), \pm\tfrac{1}{2}(e_7 - e_4 + e_5 - e_6).
\end{aligned} \tag{5.6}$$

The E_6 roots must be orthogonal to the SU(3) roots. An immediate check shows that the half-integral quaternions orthogonal to those in (5.6) are the ones which do not include e_2 and should be in the form

$$\pm[0, \tfrac{1}{2}(1 + e_1)] \qquad \pm[0, \tfrac{1}{2}(1 + e_3)], \pm[0, \tfrac{1}{2}(e_1 - e_3)]. \tag{5.7}$$

It is obvious from figure 5 that E_6 includes SO(8) as a subgroup. Therefore 24 integral quaternions are also roots of E_6. Since the weights of 8_v, 8_s and 8_c will be put on the left of the brackets they are orthogonal to the roots in (5.6) by definition. Therefore by pairing the weight of 8_v, 8_s and 8_c with those in (5.7) we get the remaining roots of E_6. Therefore 72 non-zero roots of E_6 will be as follows:

$$\pm 1, \quad \pm e_1, \quad \pm e_2, \quad \pm e_3, \quad \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \tag{5.8a}$$

$$A=[A_1, \pm\tfrac{1}{2}(1+e_1)], \qquad B=[A_3, \pm\tfrac{1}{2}(e_1-e_3)], \qquad C=[A_2, \pm\tfrac{1}{2}(1+e_3)] \tag{5.8b}$$

where A_1, A_3, A_2 are given by (3.12). The 48 roots in (5.8b) can be split as two sets of 24 roots, one with a (+) and the other with a (−) sign. If we add the 24 roots in (5.8b) with (+) signs to those in (5.8a) we obtain the roots of F_4, a subgroup of E_6. All roots of F_4 are of the same length in this case because they are embedded in E_6. Those 24 roots with (−) signs in (5.8b) represent the non-zero roots of the 26-dimensional representation of F_4. When the roots of E_6 (5.8a, b) and the roots of SU(3) in (5.6) are subtracted from (4.14) the remaining integral octonions are the weights of the representation $(\mathbf{27},\mathbf{3})+(\mathbf{27}^*,\mathbf{3}^*)$. These roots will be discussed at the end of this section and in appendix 1.

A point of main interest is the symmetries of the E_6 and SU(3) roots under the action of the elements of the binary tetrahedral group which constitute also some part of E_6. Using (3.11a) and (4.20) we can show that (5.6) and (5.8a, b) are invariant under the quaternion group. This is not the only symmetry in this case. Indeed $E_6\times SU(3)$ has much more symmetry compared with the other maximal subgroups. Now we can show that (5.6) and (5.8) are invariant under the Abelian subgroup generated by the element V. Let us consider the SU(3) roots. Since (5.6) is in the form $[0, N]$, under the action of V it transforms as

$$V: \quad [0, N]\to[0, \bar{V}^2 N]=[0, -VN]. \tag{5.9}$$

In an explicit form we have

$$\begin{aligned} V:&\quad [0, \pm e_2]\to[0, \pm\tfrac{1}{2}(1-e_1-e_2-e_3)]\\ V:&\quad [0, \pm\tfrac{1}{2}(1-e_1-e_2-e_3)]\to[0, \pm\tfrac{1}{2}(1-e_1+e_2-e_3)]\\ V:&\quad [0, \pm\tfrac{1}{2}(1-e_1+e_2-e_3)]\to[0, \pm e_2]. \end{aligned} \tag{5.10}$$

It is understood from this that the roots of SU(3) are rotated in the cyclic order under V so they constitute an invariant set. Now let us apply V on the roots of E_6. Equation (5.8a) is invariant under V as it is the binary tetrahedral group. To see how V acts on (5.8b) we use (4.20), (3.11b) and table 1 again. We can show that the set of roots in (5.8b) transform as

$$V: \quad A\to B\to C\to A. \tag{5.11}$$

It is amusing that the set of roots of E_6 rotates in a cyclic order just like the SU(3) roots. Since the whole set of roots of E_8 is invariant under any transformation of the binary tetrahedral group the roots in the coset $(\mathbf{27},\mathbf{3})+(\mathbf{27}^*,\mathbf{3}^*)$ must also remain invariant under V.

As we have noted in the case of $E_7\times SU(2)$, SU(3) can also be represented by four different sets of roots each containing one quaternionic unit $\pm e_a$ ($a=0,1,2,3$). If we wish to construct another set of roots for $E_6\times SU(3)$ we may start with, say, $\pm e_3$ instead of $\pm e_2$ and apply V on $\pm e_3$ in the same way we have done in (5.10) and obtain another V-invariant SU(3) root system. Then we construct the E_6 roots in turn in a similar fashion as described in (5.7). By following this procedure we obtain four possible decompositions of the root lattice of E_8 with respect to $E_6 \times SU(3)$; in each case V-invariance is perserved. Had we required the invariances under S, T or U in such a decomposition we would also have been able to do so. In the latter cases one selects the quaternionic unit $\pm e_a$ as one of the simple root of SU(3) and apply any one of S, T or U to get the S, T, U-invariant SU(3) root systems. Following the same procedure for V we may obtain S, T, U-invariant E_6 root systems. In each case there are always four possibilities. But one should notice that there does not exist two different invariances in a given decomposition. To summarise, there are 16 different ways of decomposing the root system of E_8 under $E_6 \times SU(3)$. In each case the action of the quaternion group leaves the decomposition invariant. We have given 16 different decompositions of integral octonions in appendix 1.

Now we come back to the magic square obtained by matching the root lattices of SU(3), SP(3) and F_4. We have already shown that $[F_4, F_4]$, $[F_4, SP(3)]$ and $[SP(3), SP(3)]$ give the root lattices of E_8, E_7 and SO(12) respectively. Now we construct the roots of SU(3) using half-integral quaternions of length one when each root is multiplied by $\sqrt{2}$. It is not possible to construct SU(3) root system either by Gaussian integers or 'half-integral' complex numbers of (3.1a, b). Therefore we should use a subset of 'half-integral' quaternions. Let us choose the simple roots of SU(3) as $\frac{1}{2}(1+e_1)$ and $\frac{1}{2}(e_2-e_1)$. The root system derived from these simple roots is given by

$$\pm\tfrac{1}{2}(1+e_1), \quad \pm\tfrac{1}{2}(e_2-e_1), \quad \pm\tfrac{1}{2}(1+e_2). \tag{5.12}$$

Now we take two sets of roots of (5.12) and pair them in the usual way. Since this pairing will furnish a root system in integral octonions it should remain in (4.14). Then we have only one possibility of pairing two such sets in the following form:

$$\begin{aligned} &[\pm\tfrac{1}{2}(1+e_1), \pm\tfrac{1}{2}(1+e_1)] \qquad [\pm\tfrac{1}{2}(e_2-e_1), \pm\tfrac{1}{2}(1+e_2)] \\ &[\pm\tfrac{1}{2}(1+e_2)], \pm\tfrac{1}{2}(e_2-e_1)]. \end{aligned} \tag{5.13}$$

These 12 integral octonions represent the root system of $SU(3) \times SU(3)$. What remains is a pairing of the roots of SU(3) with those of SP(3) with which the magic square will be completed. Thus we pair (5.3) with (5.12) provided the integral octonions so obtained remain in (4.14). In this case we are free to put (5.3) or (5.12) on the left of the bracket which will lead to two independent representations of the emerging root system. If we prefer (5.3) on the left and (5.12) on the right we obtain

$$\begin{aligned} &[(\pm 1, \pm e_1, \pm e_3), 0] = (\pm 1, \pm e_1, \pm e_3) \\ &[\tfrac{1}{2}(\pm 1 \pm e_1), \pm\tfrac{1}{2}(1+e_1)] \\ &[\tfrac{1}{2}(\pm e_1 \pm e_3), \pm\tfrac{1}{2}(e_2-e_1)] \\ &[\tfrac{1}{2}(\pm 1 \pm e_3), \pm\tfrac{1}{2}(1+e_2)]. \end{aligned} \tag{5.14}$$

These integral octonions are nothing but $6+8\times 3=30$ non-zero roots of SU(6). To summarise, we obtained the GNORS magic square in terms of integral octonions.

	SU(3)	SP(3)	F_4
SU(3)	$SU(3)\times SU(3)$	SU(6)	E_6
SP(3)	SU(6)	SO(12)	E_7
F_4	E_6	E_7	E_8

In addition to this magic square we have also found

$$[SU(2), SU(2)] = SO(4) \qquad [SP(2), SP(2)] = SO(8) \qquad [SO(9), SO(9)] = SO(16).$$

5.4. SU(9)

The Coxeter–Dynkin diagram of SU(9) can be obtained from figure 5 by deleting the root e_3 corresponding to l_6+l_7 in (4.17). Then 72 non-zero roots of the adjoint representation of SU(9) are given by the integral octonions of the form

$$l_i - l_j \qquad (i \neq j = 0, 1, \ldots, 8) \tag{5.15}$$

and the remaining 168 roots of E_8 correspond to the weights of $84+84^*$ representations of SU(9). They are given by

$$\pm(l_i + l_j + l_k - l_0) \qquad (i \neq j \neq k = 0, 1, \ldots, 8) \tag{5.16}$$

We have not seen any symmetry leaving the root lattice of SU(9) invariant. This is obvious since one element e_3 of the binary tetrahedral group is left out of the root lattice of SU(9).

5.5. SU(5)×SU(5)

In figure 5 we delete the root e_1 to obtain two disconnected Coxeter–Dynkin diagrams of SU(5)×SU(5). No symmetry is left preserving the root system of SU(5)×SU(5) since e_1 is taken out of the group structure of the binary tetrahedral group. The adjoint of SU(5)×SU(5) is represented by $20+20=40$ integral octonions as follows:

$$\begin{aligned}
[20, 0]: \quad & \pm \left[\tfrac{1}{2}(1+e_1) \quad \begin{matrix} \tfrac{1}{2}(e_3-e_2) \\ -\tfrac{1}{2}(e_3+e_2) \\ \tfrac{1}{2}(1-e_1) \\ \tfrac{1}{2}(1+e_1) \end{matrix} \right] \\
& \pm \left[0 \quad \begin{matrix} \tfrac{1}{2}(1+e_1+e_2-e_3) \\ \tfrac{1}{2}(1+e_1-e_2-e_3) \\ \tfrac{1}{2}(1-e_1+e_2-e_3) \\ \tfrac{1}{2}(1-e_1-e_2-e_3) \end{matrix} \right] \\
& [0, \pm e_1] \qquad [0, \pm e_2]
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
[0, 20]: \quad & \pm \left[\begin{matrix} \tfrac{1}{2}(e_3-e_1) \\ -\tfrac{1}{2}(e_3+e_1) \\ \tfrac{1}{2}(-1+e_2) \\ -\tfrac{1}{2}(1+e_2) \end{matrix} \quad \tfrac{1}{2}(1+e_3) \right] \\
& \pm e_2, \quad \pm e_3, \quad \pm\tfrac{1}{2}(1-e_1 \pm e_2 \pm e_3).
\end{aligned} \tag{5.18}$$

The remaining integral octonions of (4.14) transform as $(\mathbf{5}, \mathbf{10}^*)+(\mathbf{5}^*, \mathbf{10})+(\mathbf{10}, \mathbf{5})+(\mathbf{10}^*, \mathbf{5}^*)$ under SU(5)×SU(5).

5.6. $F_4 \times G_2$

This is a special subgroup of E_8. F_4 and G_2 are also exceptional groups with the following properties.

G_2 is the automorphism group of octonions and F_4 is the automorphism group of the exceptional Jordan algebra of 3×3 octonionic Hermitian matrices. G_2 is particularly interesting because one of its discrete subgroup leaves the integral octonions invariant. This point deserves further investigation; however, it will not be discussed here. Our main purpose here is to decompose 240 integral octonions under $F_4 \times G_2$.

As we have explained before we may split (5.8*b*) into two sets of 24 integral octonions. Equation (5.8*b*) can be written in the form $[a, \pm b]$ where a represents the totality of A_1, A_2 and A_3 on the left and b those on the right. If we combine $[a, b]$ with integral octonions of (5.8*a*) we obtain 48 roots of F_4. The roots of G_2 are found by adding $\pm e_4$, $\pm e_6$, $\pm e_7$ to the roots of SU(3) in (5.6). Hence the roots of G_2 are

$$\pm e_4, \quad \pm e_5, \quad \pm e_6, \quad \pm e_7, \quad \pm\tfrac{1}{2}(e_7 - e_4 - e_5 - e_6), \quad \pm\tfrac{1}{2}(e_7 - e_4 + e_5 - e_6). \tag{5.19}$$

The remaining roots of E_8 can be classified as the representations of $SO(8)\times SU(3)$ a subgroup of $F_4 \times G_2$:

$$\begin{aligned}
(\mathbf{8}_v, \mathbf{3}) + (\mathbf{8}_v, \mathbf{3}^*)&: \quad [A_1, \pm\tfrac{1}{2}(1 - e_1)] \\
& [A_1, \pm\tfrac{1}{2}(\pm e_2 \pm e_3)] && (5.20a) \\
(\mathbf{8}_s, \mathbf{3}) + (\mathbf{8}_s, \mathbf{3}^*)&: \quad [A_3, \pm\tfrac{1}{2}(e_1 + e_3)] \\
& [A_3, \tfrac{1}{2}(\pm 1 \pm e_2)] && (5.20b) \\
(\mathbf{8}_c, \mathbf{3}) + (\mathbf{8}_c, \mathbf{3}^*)&: \quad [A_2, \pm\tfrac{1}{2}(1 - e_3)] \\
& [A_2, \pm\tfrac{1}{2}(\pm e_1 \pm e_2)] && (5.20c) \\
2[(\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{3}^*)]&: \quad \pm\tfrac{1}{2}(e_7 + e_4 + e_5 + e_6), \quad \pm\tfrac{1}{2}(e_7 + e_4 + e_5 - e_6) \\
& \pm\tfrac{1}{2}(e_7 - e_4 + e_5 + e_6), \quad \pm\tfrac{1}{2}(e_7 - e_4 - e_5 + e_6) \\
& \pm\tfrac{1}{2}(e_7 + e_4 - e_5 + e_6), \quad \pm\tfrac{1}{2}(e_7 + e_4 - e_5 - e_6). && (5.20d)
\end{aligned}$$

There are $3\times 48 + 4\times 3 = 156$ integral octonions in (5.20*a, b, c, d*). When we add this to 24 non-zero roots in the form $[a, -b]$ of (5.8*b*) we obtain 180 roots of the coset space $E_8/F_4\times G_2$. If this number is added to $48+12$ non-zero roots of F_4 and G_2 we get back 240 integral octonions of E_8. This decomposition of E_8 is also invariant under the quaternion group.

6. Discussions and conclusion

The method we have described for the constructions of the root systems of SO(4), SP(2), SO(8), F_4 and E_8 from that of SU(2) is very simple and perhaps useful. A pair of elements $[a, b]$ belonging to a division algebra furnished with a product rule is independent of the particular choices of the imaginary units of the corresponding division algebras. Therefore it provides a unified way of treating the root systems of the corresponding Lie algebras. This is particularly useful in the case of E_8 since the choices of the imaginary units of octonions are somewhat arbitrary.

The root lattices expressed in terms of integral elements of the corresponding division algebras have overwhelming properties, namely they correspond to certain discrete groups or algebras. These aspects single out the groups SU(2), $SU(2)\times SU(2)$, SO(8) and E_8 and also SP(2), SO(9) and F_4 slightly less so. One feature of this method is the possibility of classification of the maximal subgroups of E_8 under the actions of the subsets of the discrete subgroup of SU(2) of 24 integral quaternions.

We have also noted that the root system of F_4 is invariant under the binary tetrahedral group. There exists an Abelian subgroup of the binary tetrahedral group generated by an element S which has been used by GNORS to determine the cocyles arising in the vertex construction of a level one representation of the affine F_4 algebra. It is perhaps more interesting to use the full group invariance in such a problem and check what effect will result.

Another interesting aspect is the classification of the embeddings of $E_6 \times SU(3)$ in E_8 respecting various invariances of Abelian subgroups generated by S, T, U and V. This may have some connections with orbifold compactifications of the heterotic string if the discrete subgroups of SU(2) with 24 or 48 elements are embedded in one of its maximal subgroups of E_8.

We have also constructed the root lattices of the groups of the GNOR magic square in terms of integral octonions. Integral elements of division algebras are also proposed for a new description of lattice gauge fields in which the fundamental object is a discrete version of a principle fibre bundle [18].

The roots of E_8 in the form $[A_1, A_1]$ in (4.5) is the $(\mathbf{8}_v, \mathbf{8}_v)$ representation of $SO(8) \times SO(8)$. It was clear from the discussion following (4.9f) that all the roots of E_8 can be generated by squaring $[A_1, A_1]$. There is a close correspondence between this structure and the construction of the level one representation of the affine E_8 algebra [19].

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Appendix 1. Invariant decompositions of the E_8 root lattice under $E_6 \times SU(3)$

As we have explained in § 5 there are four different embeddings of $E_6 \times SU(3)$ in E_8. In each embedding one of the Abelian subgroups S, T, U and V of the binary tetrahedral group is preserved. In each embedding there are four possible choices for the roots of SU(3) and correspondingly for E_6. We give one invariant embedding in an explicit form and explain how it should be understood. Since $SO(8) \subset E_6$ and $SU(3) \subset SO(8)$ we take the advantage of pairing of the weights of 8_v, 8_s, 8_c with the 'half integral' quaternions orthogonal to the SU(3) roots. In table 2 we give S invariant embeddings of $E_6 \times SU(3)$ in E_8.

Table 2 should be understood as follows. If the roots of SU(3) are chosen as

$$[0, \pm 1] = \pm e_7 \qquad [0, \pm S] = \pm \tfrac{1}{2}(e_7 + e_4 + e_5 + e_6)$$
$$[0, \pm \bar{S}] = \pm \tfrac{1}{2}(e_7 - e_4 - e_5 - e_6) \tag{A1.1}$$

then the corresponding E_6 roots are

$$\pm 1, \quad \pm e_1, \quad \pm e_2, \quad \pm e_3, \quad \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$$
$$[A_1, \pm\tfrac{1}{2}(e_2 - e_3)], \quad [A_2, \pm\tfrac{1}{2}(e_1 - e_2)], \quad [A_3, \pm\tfrac{1}{2}(e_3 - e_1)] \tag{A1.2}$$

where A_1, A_2 and A_3 are given by (3.12). The roots of E_8 in the representations $(\mathbf{27}, \mathbf{3}) + (\mathbf{27^*}, \mathbf{3^*})$ of $E_6 \times SU(3)$ are obtained by pairing the elements in columns labelled by 1, 2, 3 with the corresponding elements in the first unlabelled elements, i.e. with three zeros in the case of SU(3) roots and with A_1, A_2 and A_3 in the case of E_6 roots. Under the action of S, the roots in (A1.1) and (A1.2) are rotated in cyclic order from

Table 2. S-invariant embeddings of $E_6 \times SU(3)$ in E_8.

	0	1	2	3
SU(3) roots:	$[0, \pm 1,$	$\pm e_1,$	$\pm e_2,$	$\pm e_3]$
	$[0, \pm S,$	$\pm U,$	$\pm \bar{T},$	$\pm \bar{U}]$
	$[0, \pm \bar{S},$	$\pm \bar{V},$	$\pm V,$	$\pm T]$
E_6 roots:	$[A_1, \pm\frac{1}{2}(e_2 - e_3), \pm\frac{1}{2}(e_2 + e_3), \pm\frac{1}{2}(1 - e_1), \pm\frac{1}{2}(1 + e_1)]$			
	$[A_2, \pm\frac{1}{2}(e_1 - e_2), \pm\frac{1}{2}(1 - e_3), \pm\frac{1}{2}(1 + e_3), \pm\frac{1}{2}(e_1 + e_2)]$			
	$[A_3, \pm\frac{1}{2}(e_3 - e_1), \pm\frac{1}{2}(1 + e_2), \pm\frac{1}{2}(e_3 + e_1), \pm\frac{1}{2}(1 - e_2)]$			
	$\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$			

Table 3. T-invariant embeddings of $E_6 \times SU(3)$ in E_8.

	0	1	2	3
SU(3) roots:	$[0, \pm 1,$	$\pm e_1,$	$\pm e_2,$	$\pm e_3]$
	$[0, \pm T,$	$\pm V,$	$\pm U,$	$\pm \bar{S}]$
	$[0, \pm \bar{T},$	$\pm \bar{U},$	$\pm S,$	$\pm \bar{V}]$
E_6 roots:	$[A_1, \pm\frac{1}{2}(e_2 - e_3), \pm\frac{1}{2}(e_2 + e_3), \pm\frac{1}{2}(1 - e_1), \pm\frac{1}{2}(1 + e_1)]$			
	$[A_3, \pm\frac{1}{2}(e_3 + e_1), \pm\frac{1}{2}(1 - e_2), \pm\frac{1}{2}(e_1 - e_3), \pm\frac{1}{2}(1 + e_2)]$			
	$[A_2, \pm\frac{1}{2}(e_1 + e_2), \pm\frac{1}{2}(1 + e_3), \pm\frac{1}{2}(1 - e_3), \pm\frac{1}{2}(e_1 - e_2)]$			
	$\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$			

Table 4. U-invariant embeddings of $E_6 \times SU(3)$ in E_8.

	0	1	2	3
SU(3) roots:	$[0, \pm 1,$	$\pm e_1,$	$\pm e_2,$	$\pm e_3]$
	$[0, \pm U,$	$\pm \bar{S},$	$\pm T,$	$\pm V]$
	$[0, \pm \bar{U},$	$\pm \bar{T},$	$\pm \bar{V},$	$\pm S]$
E_6 roots:	$[A_1, \pm\frac{1}{2}(e_2 + e_3), \pm\frac{1}{2}(e_2 - e_3), \pm\frac{1}{2}(1 + e_1), \pm\frac{1}{2}(1 - e_1)]$			
	$[A_3, \pm\frac{1}{2}(e_3 - e_1), \pm\frac{1}{2}(1 + e_2), \pm\frac{1}{2}(e_3 + e_1), \pm\frac{1}{2}(1 - e_2)]$			
	$[A_2, \pm\frac{1}{2}(e_1 + e_2), \pm\frac{1}{2}(1 + e_3), \pm\frac{1}{2}(1 - e_3), \pm\frac{1}{2}(e_1 - e_2)]$			
	$\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$			

Table 5. V-invariant embeddings of $E_6 \times SU(3)$ in E_8.

	0	1	2	3
SU(3) roots:	$[0, \pm 1,$	$\pm e_1,$	$\pm e_2,$	$\pm e_3]$
	$[0, \pm V,$	$\pm T,$	$\pm \bar{S},$	$\pm U]$
	$[0, \pm \bar{V},$	$\pm S,$	$\pm \bar{U},$	$\pm \bar{T}]$
E_6 roots:	$[A_1, \pm\frac{1}{2}(e_2 + e_3), \pm\frac{1}{2}(e_2 - e_3), \pm\frac{1}{2}(1 + e_1), \pm\frac{1}{2}(1 - e_1)]$			
	$[A_3, \pm\frac{1}{2}(e_3 + e_1), \pm\frac{1}{2}(1 - e_2), \pm\frac{1}{2}(e_3 - e_1), \pm\frac{1}{2}(1 + e_2)]$			
	$[A_2, \pm\frac{1}{2}(e_1 - e_2), \pm\frac{1}{2}(1 - e_3), \pm\frac{1}{2}(1 + e_3), \pm\frac{1}{2}(e_1 + e_2)]$			
	$\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)$			

top to bottom so that each column remains invariant under S. Similarly the elements in columns 1, 2, 3 are rotated to each other in their own sectors. Therefore coset structure is preserved by S. Thus whenever one column is preferred for the roots for SU(3) and E_6 the other columns serve for the coset space. This displays the four different S-invariant decompositions of E_8 under $E_6 \times SU(3)$. Similar tables for T, U and V can be made as in tables 3, 4 and 5.

Appendix 2. Seven modules of integral octonions

In this appendix we explain how seven distinct root systems of E_8 each of which consists of 240 integral octonions can be constructed. Each module is closed under multiplication. We have already shown in (4.12) that 240 integral octonions describing the root lattice of E_8 are given by

$$240 = [A_0, 0] + [0, A_0] + [A_1, A_1] + [A_2, A_3] + [A_3, A_2] \tag{A2.1}$$

where A_i $(i = 0, 1, 2, 3)$ are defined by (3.12) and (4.5). This root system corresponds to the Coxeter–Dynkin diagram of E_8 in figure 5. Since labelling of the vector, spinor and antispinor representations of SO(8) is arbitrary we could have started with a labelling of the vector representation by the set of weights A_2 and/or A_3. Then this would lead to two different sets of integral octonions each of which is equally describing the root lattice of E_8. They would be given by

$$240 = [A_0, 0] + [0, A_0] + [A_2, A_2] + [A_3, A_1] + [A_1, A_3] \tag{A2.2}$$

$$240 = [A_0, 0] + [0, A_0] + [A_3, A_3] + [A_1, A_2] + [A_2, A_1]. \tag{A2.3}$$

These sets of integral octonions correspond to different assignments of the simple roots in figure 5. Equation (A2.2), e.g., can be obtained from a diagram where the two diagrams of SO(8) in figure 4 are rotated, one in the clockwise direction and the other in the counterclockwise direction such that e_5 and e_2 are connected to a root of the form $\frac{1}{2}(-1 - e_2 - e_5 - e_7) = -[\frac{1}{2}(1 + e_2), \frac{1}{2}(1 + e_2)]$. To obtain a similar diagram of E_8 we delete the root e_4 in the new diagram and add the root $\frac{1}{2}(-1 - e_3 - e_4 - e_7) = -[\frac{1}{2}(1 + e_3), \frac{1}{2}(1 + e_1)]$ to the right end of the diagram which will be connected to e_3. For (A2.3) a similar procedure can be followed. We should note that the roots $[A_0, 0]$ and $[0, A_0]$ are unaltered under these changes. We also note that a term $[M, N]$ in (A2.1) and (A2.3) should be understood as $[M, N] = M + e_7 N$.

The remaining four modules of 240 integral octonions can be obtained replacing associative triad $e_1 e_2 e_3$ by other triads. Since there are seven associative triads 123, 246, 435, 367, 651, 572, 714 one can start with any one of them. Let us choose the triads involving e_1 besides $e_1 e_2 e_3$. Then we obtain the remaining four modules. Actually we obtain three modules for 651 and three modules for 714; however, in each case one module coincides with the one already constructed. When we choose e_6, e_5, e_1 as our new quaternionic units with $e_4 e_6 = e_2$, $e_4 e_5 = e_3$, $e_4 e_1 = e_7$ we first construct SO(8) roots and eight-dimensional representations with e_6, e_5 and e_1. Then we construct integral octonions with new quaternionic units in which case e_4 plays the role of independent imaginary unit. Let us denote A_0' representing 24 integral quaternions generated by e_6, e_5, e_1:

$$A_0' \colon \ \pm 1, \pm e_6, \pm e_5, \pm e_1, \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_5 \pm e_6). \tag{A2.4}$$

Let the vector, spinor and antispinor representations of SO(8) be given by the sets of weights

$$\begin{array}{ccc} A_6 & A_1 & A_5 \\ \hline \frac{1}{2}(\pm 1 \pm e_6) & \frac{1}{2}(\pm 1 \pm e_1) & \frac{1}{2}(\pm 1 \pm e_5) \\ \frac{1}{2}(\pm e_5 \pm e_1) & \frac{1}{2}(\pm e_6 \pm e_5) & \frac{1}{2}(\pm e_1 \pm e_6). \end{array} \tag{A2.5}$$

With (A2.4) and (A2.5) we can construct the following three sets of modules of integral octonions:

$$240 = [A_0', 0] + [0, A_0'] + [A_6, A_6] + [A_1, A_5] + [A_5, A_1] \tag{A2.6}$$

$$240 = [A_0', 0] + [0, A_0'] + [A_5, A_5] + [A_6, A_1] + [A_1, A_6] \tag{A2.7}$$

$$240 = [A_0', 0] + [0, A_0'] + [A_1, A_1] + [A_5, A_6] + [A_6, A_5]. \tag{A2.8}$$

Here (A2.8) coincides with (4.14) so that this module is not independent. We note that the brackets in (A2.6, 7, 8) should be understood as follows:

$$[M, N] = M + e_4 N.$$

In the case of the quaternionic units (e_7, e_1, e_4), e_5 is the independent imaginary unit satisfying

$$e_5 e_7 = e_2 \qquad e_5 e_1 = e_6 \qquad e_5 e_4 = e_3.$$

In this case we denote 24 integral quaternions by A_0'' given by

$$A_0'': \quad \pm 1, \quad \pm e_7, \quad \pm e_1, \quad \pm e_4, \quad \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_4 \pm e_7). \tag{A2.9}$$

The weights of vector, spinor and antispinor representation can be written as

$$\begin{array}{ccc} A_7 & A_4 & A_1 \\ \hline \frac{1}{2}(\pm 1 \pm e_7) & \frac{1}{2}(\pm 1 \pm e_4) & \frac{1}{2}(\pm 1 \pm e_1) \\ \frac{1}{2}(\pm e_1 \pm e_4) & \frac{1}{2}(\pm e_7 \pm e_1) & \frac{1}{2}(\pm e_4 \pm e_7). \end{array} \tag{A2.10}$$

Using (A2.9) and (A2.10) we construct three more modules of integral octonions two of which are independent:

$$240 = [A_0'', 0] + [0, A_0''] + [A_7, A_7] + [A_4, A_1] + [A_1, A_4] \tag{A2.11}$$

$$240 = [A_0'', 0] + [0, A_0''] + [A_1, A_1] + [A_7, A_4] + [A_4, A_7] \tag{A2.12}$$

$$240 = [A_0'', 0] + [0, A_0''] + [A_4, A_4] + [A_1, A_7] + [A_7, A_1]. \tag{A2.13}$$

Here (A2.12) coincides with (4.14), therefore it is not independent. The brackets here should be understood as $[M, N] = M + e_5 N$.

Therefore seven independent modules of integral octonions are (A2.1), (A2.2), (A2.3), (A2.6), (A2.7), (A2.11) and (A2.13). We have already noted that these modules can also be obtained replacing the imaginary units in (A2.1) by successive applications of the changes of the units in the cyclic order $1 \to 2 \to 4 \to 3 \to 6 \to 5 \to 7 \to 1$. In each module one of the imaginary unit plays an essential role by representing the vector representations $(\mathbf{8}_v, \mathbf{8}_v)$ in the form $[A_i, A_i]$ $(i = 1, 2, \ldots, 7)$ in a given module.

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